The Good, The Bad, and The Ugly: An Inquiry into the Causes and Nature of Credit Cycles

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Abstract

This paper builds models of nonlinear dynamics in the aggregate investment and borrower net worth to study the causes and nature of endogenous credit cycles. The basic model has two types of projects: *the Good* and *the Bad*. The Good projects rely on the inputs supplied by others who could undertake investment in the future, thereby improving their net worth. The Bad projects are independently profitable so that they do not improve the net worth of other borrowers. Furthermore, they are subject to the borrowing constraint due to some agency problems. With a low net worth, the agents cannot finance the Bad, and much of the credit goes to finance the Good, even when the Bad projects are more profitable than the Good projects. This over-investment to the Good creates a boom, leading to an improvement in borrower net worth. This makes it possible for the agents to finance the Bad. This shift in the composition of the credit from the Good to the Bad at the peak of the boom causes a deterioration of borrower net worth. The whole process repeats itself. Endogenous fluctuations occur, as the Good breed the Bad and the Bad destroy the Good.

The model is then extended to add a third type of projects, *the Ugly*, which are unproductive but subject to no borrowing constraint. With a low net worth, the Good compete with the Ugly, which act as a drag on the Good, creating the *credit multiplier* effect. With a high net worth, the Good compete with the Bad, which destroy the Good, creating the *credit reversal* effect. By combining these two effects, this hybrid model generates *intermittency* phenomena, i.e., relatively long periods of small and persistent movements punctuated intermittently by seemingly random-looking behaviors. Along these cycles, the economy exhibits asymmetric fluctuations; it experiences a slow process of recovery from a recession, followed by a rapid expansion, and possibly after a period of high volatility, plunges into a recession.

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1. Introduction

It is commonly argued that an economic expansion often comes to an end as a result of the changing nature of credit and investment at the peak of the boom. According to the popular argument, "success breeds crises." After prolonged periods of expansion, more credit becomes extended to finance some "questionable" activities. Such an extension of credit causes volatility and destabilizes the economy.[2] Central bankers indeed seem concerned that financial frenzies that emerge after a period of economic expansion might lead to misallocation of credit, thereby pushing the economy into a recession, and they often attempt to take precautionary measures to cool down the boom and to achieve a soft landing of the economy.

This paper develops dynamic general equilibrium models of endogenous credit cycles, which provide a theoretical support for the view that changing compositions of credit and of investment are responsible for creating instability and fluctuations. Furthermore, the equilibrium dynamics display some features reminiscent of the popular argument. Contrary to the popular argument, however, the agents are assumed to be fully rational and instability is not caused by "euphoria," "manias," or "irrational exuberance." Indeed, fluctuations are not at all driven by the expectations of the agents, whether they are rational or not. In the models developed below, the equilibrium path is unique, and the cycles are purely deterministic. Endogenous fluctuations occur when the unique steady state of the time-invariant, deterministic nonlinear dynamical system loses its stability. They are based on neither 'sunspots" nor "bubbles," nor any form of indeterminacy or self-fulfilling expectations.[3]

Behind instability in our models is the heterogeneity of investment projects. Investment projects differ in many dimensions. They differ not only in profitability. They differ also in the severity of agency problems, which determine their borrowing constraints. In addition, they differ in the input requirements, which create different general equilibrium effects, different degrees of demand spillovers, or "backward linkages," to use Hirschman (1958)'s terminology. As a result, not all the profitable investments contribute equally to the overall balance sheet condition of the economy.

[2]Kindleberger (1996, Ch.2) offers a lucid exposition of the popular argument, the most well-known of which is the financial instability hypothesis of Minsky (1975;1982). See also Hawtrey (1913) for an earlier example.
[3]For broad surveys on endogenous cycles, Boldrin and Woodford (1990) and Guesnerie and Woodford (1992).

For example, imagine that there are two types of profitable investment projects, which we shall call *the Good* and *the Bad*. The Good projects rely on the inputs supplied by others who could undertake investment in the future. By generating demand for these inputs, the Good projects improve the net worth of those who supply these inputs. The Bad are independently profitable so that they do not require the inputs supplied by others and hence fail to improve their net worth. In addition, the Bad projects are subject to borrowing constraints due to some agency problems. When the net worth is low, the agents are unable to finance the Bad projects, and much of the credit goes to finance the Good projects, even when the Bad projects may be more profitable than the Good projects. This over-investment to the Good projects generates high demand spillovers, creating a boom and leading to an improvement in borrower net worth. During a boom, with an improved net worth, the agents are now able to finance the Bad projects. The credit is now redirected from the Good to the Bad. This change in the composition of credit and of investment at the peak of the boom causes a deterioration of borrower net worth. The whole process repeats itself. Along these cycles, the Good breed the Bad and the Bad destroy the Good, as in ecological cycles driven by predator-prey or host-parasite interactions.[4] We call these two types of projects the Good and the Bad, not because of their welfare implications. We call them the Good and the Bad, because of their different propensity to generate wealth for other investors. Key for generating instability and endogenous fluctuations are: a) some profitable investments contribute less to improve net worth of other borrowers; and b) such investments are subject to agency problems, which are neither too big nor too small, so that the agents can finance them when their net worth is sufficiently high, but not when their net worth is low.

Many recent studies in macroeconomics of credit market frictions have investigated the role of borrower net worth in the propagation mechanisms; see Matsuyama (2008) for a survey. Among the most influential is Bernanke and Gertler (1989). Their study, as well as many others, focused on the *credit multiplier* mechanism: how the borrowing constraints introduce *persistence* into the aggregate investment dynamics. In the absence of exogenous shocks, there would be no

[4]While the intuition behind fluctuations is similar with that of predator-prey cycles in biology, our models are quite different from what mathematical biologists call the predator-prey models (see, e.g., Murray 1990).

recurrent fluctuations in their model.[5] The present study, on the other hand, emphasizes the *credit reversal* mechanism: how borrowing constraints introduce *instability* into the dynamics, which causes recurrent fluctuations even in the absence of any external shock. It should be pointed out that the present study and Bernanke-Gertler both share the observation that, in the presence of credit market frictions, saving does not necessarily flow into the most profitable investment projects, and that this problem can be alleviated (aggravated) by a higher (lower) borrower net worth. The two studies differ critically in the assumption on the set of profitable investment projects that compete for credit. In the Bernanke and Gertler model, all the profitable investments contribute equally to improve net worth of other borrowers. It is assumed that the only alternative use of saving in their model, storage, is unprofitable, subject to no borrowing constraint, and generates no demand spillovers. This means that, when an improved net worth allows more saving to flow into the profitable investments, saving is redirected *towards* the investments that generate demand spillovers, which further improve borrower net worth. This is the mechanism behind the credit multiplier effect in their model (and many others in the literature). The present study differs from Bernanke and Gertler in that *not all* the profitable investments have the same demand spillover effects. *Some* profitable investments, which are subject to the borrowing constraints, do not improve the net worth of other borrowers. This means that, when an improved net worth allows more saving to flow into such profitable investments, saving may be redirected *away from* the investments that generate demand spillovers, which causes a deterioration of borrower net worth. This is the mechanism behind the credit reversal effect.

Needless to say, these two mechanisms are not mutually exclusive and can be usefully combined. We will indeed present a hybrid model, which allows for three types of projects, *the Good*, *the Bad*, and *the Ugly*. Only the Good improve the net worth of other borrowers; neither the Bad nor the Ugly improve net worth of other borrowers. The Bad are profitable but subject to the borrowing constraint. The Ugly are unprofitable but subject to no borrowing constraint (as

[5]In one variation of their models, Kiyotaki and Moore (1997; Section III) demonstrated that the equilibrium dynamics display oscillatory convergence to the steady state, which is why they called their paper, "Credit Cycles." However, these oscillations occur because they added the assumption that the investment opportunity arrives stochastically to each agent. The borrowing constraints in all of their models work only to amplify the movement

the storage technology in the Bernanke-Gertler model). Thus, when the net worth is low, the Good compete with the Ugly, which act as a drag on the Good so that the model behaves like the Bernanke-Gertler model, with its credit multiplier effect. When the net worth is high, the Good compete with the Bad, which destroy the Good, creating the credit reversal effect, as in the basic model. By combining the two effects, this hybrid model generates *intermittency* phenomena. That is to say, relatively long periods of small and persistent movements are punctuated intermittently by seemingly random-looking behaviors. Along these cycles, the economy exhibits asymmetric fluctuations; it experiences a slow process of recovery from a recession, followed by a rapid expansion, and, possibly after a period of high volatility, plunges into a recession.

A few existing studies have demonstrated endogenous credit cycles through some sorts of credit reversal mechanism. In Azariadis and Smith (1997), the source of credit friction is adverse selection; financial markets cannot tell savers from investors. With a higher capital stock, the rate of return on saving is so low that savers would have incentive to pretend to be investors. To prevent this, the credit market is characterized by a separating equilibrium, in which the volume of credit offered to borrowers is restricted, which leads to a lower investment. In Aghion, Banerjee, and Piketty (1999), demand spillovers from the investment benefit savers more than investors so that a high investment shifts wealth distribution towards savers, which makes investors more dependent on external finance, which leads to a lower investment. In Matsuyama (2007; 2008, sec. 5.1.2), cycles occur because an improved net worth shifts the credit composition towards small scale projects with lower productivity. In Matsuyama (2008, sec. 5.2), an improved net worth shifts the credit composition towards less productive projects that come with bigger private benefits.[6] None of these models generate intermittency and asymmetric fluctuations.

The rest of the paper is organized as follows. Section 2 presents the model of the Good and Bad projects, and derives the dynamical system that governs the equilibrium trajectory under the additional assumption that the Good are not subject to any borrowing constraint. Section 3 characterizes the equilibrium path for the full set of parameter values, which enables us to identify the condition under which the steady state loses its stability and endogenous fluctuations occur.

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caused by shocks, instead of reversing it. In any case, in all of their models, the steady state is stable and any fluctuations will dissipate in the absence of exogenous shocks.

[6] Matsuyama (2008, sec.5.3.2 and 5.3.3) briefly sketches a few results in this paper.

The main conclusion is that, when the Bad are sufficiently profitable, instability and fluctuations occur when agency problems for the Bad are neither too low nor too high. Section 4 reintroduces a borrowing constraint for the Good projects. Section 5 develops a model of the Good, the Bad, and the Ugly, which combines both credit multiplier and credit reversal effects and shows how intermittency and asymmetric fluctuations occur. Section 6 offers some concluding comments.

2. The Good and The Bad.

Time is discrete and extends from zero to infinity ($t = 0, 1, \ldots$). The basic framework used is the Diamond (1965) overlapping generations model with two period lives. There is one final good, the *numeraire*, which can be either consumed or invested. In each period, a unit measure of agents arrives and stay active for two periods. In the first period, each agent is endowed with and supplies inelastically one unit of the input called "labor" at the competitive "wage rate", w_t. The agents consume only in the second. Thus, the aggregate labor supply is $L_t = 1$, and the equilibrium value of their labor endowment, w_t, is also the net worth of the young at the end of period t. The young in period t need to allocate their net worth to finance their consumption in period t+1. The following options are available to them.

First, all the young agents can lend a part or all of the net worth in the competitive credit market, which earns the gross return equal to r_{t+1} per unit. If they lend the entire net worth, their second-period consumption is equal to $r_{t+1}w_t$. Second, some young agents have access to an investment project and may use a part or all of the net worth to finance it. There are two types of projects, both of which come in discrete units. Each young agent has access to at most one type of the project, and each young agent can manage at most one project. More specifically,

The Good: A fraction μ_1 of the young has access to the Good projects. To help the narrative, let us call them *entrepreneurs*, who know how to set up a firm. Setting up a firm requires one unit of the final good invested in period t. This enables these agents to produce $\phi(n_{t+1})$ units of the final good in period t+1 by employing n_{t+1} units of labor supplied by the next generation at the competitive wage rate, w_{t+1}. The production function satisfies $\phi(n) > 0$, $\phi'(n) > 0$ and $\phi''(n) < 0$ for all $n > 0$. Maximizing the profit, $\phi(n_{t+1}) - w_{t+1}n_{t+1}$, yields the demand for labor per firm, $w_{t+1} = \phi'(n_{t+1})$. The equilibrium profit from running a firm in period t+1 can thus be expressed as an increasing function of the equilibrium employment, $\pi_{t+1} = \pi(n_{t+1}) \equiv \phi(n_{t+1}) - \phi'(n_{t+1})n_{t+1}$ with $\pi'(n_{t+1}) = -\phi''(n_{t+1})n_{t+1} > 0$.

If $w_t < 1$, these agents need to borrow $1 - w_t > 0$ in the competitive credit market to start the project. If $w_t > 1$, they can start the project and lend $w_t - 1 > 0$. In either case, the second-period consumption is equal to $\pi_{t+1} - r_{t+1}(1-w_t)$ if they start the project, which is greater than or equal to $r_{t+1}w_t$ (the second-period consumption if they lend the entire net worth in the credit market) if and only if

(1) $\pi_{t+1} \geq r_{t+1}$.

The entrepreneurs are willing to set up firms if and only if the profitability condition, (1), holds.

The Bad: A fraction $\mu_2 \leq 1 - \mu_1$ of the young have access to a project, which requires m units of the final good to be invested in period t and generates Rm units of the final good in period t+1. In want of better terms, let us call them *traders*. Note that, unlike the entrepreneurs, their capital does not require the use of "labor" as the complementary input. We may thus interpret their activities as holding onto the final good for one period to earn the gross return equal to R per unit, without generating any input demand.

If $w_t < m$, these agents need to borrow $m - w_t > 0$ to start the project. If $w_t > m$, they can start the project and lend $w_t - m > 0$. Their second-period consumption is thus equal to $Rm-r_{t+1}(m-w_t)$ as a trader, which is greater than $r_{t+1}w_t$ if and only if

(2) $R \geq r_{t+1}$.

The traders are willing to start their operation if and only if (2) holds.

Remark 1: Note that the terminology, the Good and the Bad, reflects differential propensity to generate pecuniary externalities; the Good improve the net worth of future borrowers but the Bad fail to do so. Here, this key feature is introduced by assuming that the Good rely on the "labor" supplied by the next generation, while the Bad is independently profitable. "Labor" in our model should not be literally interpreted. Instead it should be interpreted more broadly to include any inputs supplied or any assets held by potential borrowers, who could sell them or use them as collaterals to ease their borrowing constraints. Beyond such differential general equilibrium prices effects, our mechanism does not require what these projects must be like. In more general settings, the projects that generate more pecuniary externalities than others need not be more "productive," more "socially beneficial" nor more "labor-intensive." Nor should the designation, "entrepreneurs" and "traders" be literally interpreted. Their identity is not essential beyond the types of projects they initiate, so that it should be interpreted merely as a mnemonic device, which helps narrative when discussing two types of agents.[7] Matsuyama (2004) discussed more extensively how these projects can be given different interpretations with different empirical implications.

Indeed, it is not essential that different agents have access to different projects. One could alternatively assume that all the agents are homogenous and have access to both types of projects. As long as no agent can invest in both projects simultaneously and the creditor can observe the type of the investment made by the borrower, the results would carry over, even though it would make the derivation of the equilibrium condition far more complicated. Nor is it essential that each agent can manage at most one project. This assumption reduces each agent's investment decision to a binary choice, simplifying the analysis, although it introduces the need for additional parameter restrictions; see (A2) and (A3) later. The assumption of the minimum investment requirement is essential. Without the nonconvexity, the borrowing constraint introduced later would never be binding.[8] The assumption that the two projects may have different minimum requirements does not play any essential role in this paper.[9]

The Borrowing Constraints:

The credit market is competitive in that both lenders and borrowers take the equilibrium rate of return, r_{t+1}, given. It is imperfect, however, in that one may not be able to borrow any amount at the equilibrium rate. The borrowing limit exists because the borrowers can pledge only up to a fraction of the project revenue for the repayment.[10] More specifically, the entrepreneurs

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[7] Alternatives that have been suggested to me include "employers vs. non-employers," "Good vs. Bad agents," or "Type-I vs. Type-II agents," but I found them rather cumbersome.
[8] This is partly due to the assumption that all the agents have the same net worth. If there are sufficient mismatches between those who own the endowment and those who have access to the projects, the borrowing constraint could be binding even when the projects are divisible.
[9] This is in contrast to Matsuyama (2007; 2008, sec.5.1.2), in which credit reversal occurs as the composition shifts towards less productive project that comes with the smaller minimum requirement.
[10] See Tirole (2005) for the pledgeability approach for modeling credit market frictions and Matsuyama (2008) for a variety of applications in macroeconomics. They also discuss various stories of agency problems that can be told to

would not be able to credibly commit to repay more than $\lambda_1\pi_{t+1}$, where $0 \le \lambda_1 \le 1$. Knowing this, the lenders would allow the entrepreneurs to borrow only up to $\lambda_1\pi_{t+1}/r_{t+1}$. Thus, the entrepreneurs can start their businesses only if

(3) $w_t \ge 1 - \lambda_1\pi_{t+1}/r_{t+1}$.

The borrowing constraint thus takes a form of the net worth requirement. The entrepreneurs set up their firms, only when both (1) and (3) are satisfied. Note that (3) implies (1) if $w_t \le 1 - \lambda_1$ and that (1) implies (3) if $w_t \ge 1 - \lambda_1$. In other words, the profitability is a relevant constraint when $w_t > 1 - \lambda_1$, while the borrowing constraint is a relevant constraint when $w_t < 1 - \lambda_1$. Likewise, the traders would not be able to credibly commit to repay more than λ_2Rm, where $0 \le \lambda_2 \le 1$. Knowing this, the lender would allow the traders to borrow only up to λ_2Rm/r_{t+1}. Thus, they cannot start their operations unless

(4) $w_t \ge m[1 - \lambda_2R/r_{t+1}]$.

The traders invest in their operations, only when both (2) and (4) are satisfied. Note that (4) implies (2) if $w_t \le (1 - \lambda_2)m$ and that (2) implies (4) if $w_t \ge (1 - \lambda_2)m$. Again, the borrowing constraint (4) can be binding only if $w_t \le (1 - \lambda_2)m$.

As it turns out, the borrowing constraint for the Good is not essential for generating the credit reversal mechanism that causes instability and fluctuations. We will therefore set $\lambda_1 = 1$ and drop the subscript from λ_2 and let $\lambda_2 = \lambda < 1$ until section 3. It will be shown in section 4 that, for any fixed $\lambda_2 < 1$, the results are robust to a small reduction in λ_1 from $\lambda_1 = 1$. Allowing $\lambda_1 < 1$ is crucial for the extension in section 5, which introduces the credit multiplier effect.

Equilibrium Wage and Business Profit:

Let $k_{t+1} \le \mu_1$ be the number of young entrepreneurs in period t that start their firms (hence it is the number of active firms in period t+1). Let $x_{t+1} \le \mu_2$ be the number of young traders in period t that start their operations (hence their total investment is equal to mx_{t+1}.) Since only the firms hire labor, the labor market equilibrium in period t+1 is $n_{t+1}k_{t+1} = 1$, from which $n_{t+1} = 1/k_{t+1}$.

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justify the assumption that the borrowers can pledge only up to a fraction of the project revenue. Nevertheless, its main appeal is the simplicity, which makes it suitable for studying general equilibrium implications of credit market imperfections. In the following discussion, the pledgeability of each project is treated as the inverse measure of its agency problem.

Thus, the equilibrium wage rate and the business profit per firm in period t+1 may be expressed as functions of k_{t+1}:

(5) $w_{t+1} = \phi'(1/k_{t+1}) \equiv W(k_{t+1})$

(6) $\pi_{t+1} = \pi(1/k_{t+1}) = \phi(1/k_{t+1}) - \phi'(1/k_{t+1})/k_{t+1} \equiv \Pi(k_{t+1})$,

where $W'(k_{t+1}) > 0$ and $\Pi'(k_{t+1}) < 0$. A higher business investment means a high wage and a lower profit. Thus, the Good generate demand for labor and drives up the wage rate, thereby improving the net worth of the next generation, unlike the Bad, which do not rely on labor. It is also straightforward to show that these functions satisfy $\phi(1/k)k = k\Pi(k) + W(k)$ and $k\Pi'(k) + W'(k) = 0$ as the identities.

In addition, we make the following assumptions.

(A1) There exists $K > 0$, such that $W(K) = K$ and $W(k) > k$ for all $k \in (0, K)$.

(A2) $K < \mu_1$.

(A3) $\max_{k \in [0, K]} \{W(k) - k\} < m\mu_2$.

(A4) $\lim_{k \to +0} \Pi(k) = +\infty$.

For example, let $\phi(n)= (Kn)^{\beta}/\beta$, with $K < \mu_1$ and $0 < \beta < 1$. Then, (A1), (A2) and (A4) are all satisfied. (A3) is also satisfied if $K < (m\mu_2)/\beta(1-\beta)^{(1-\beta)/\beta}$. (A1) is introduced only to rule out an uninteresting case, where the dynamics of k_t would converge to zero in the long run. It will be shown later that, if $k_t \in (0, K]$, $k_s \in (0, K]$ for all $s > t$, so that $W(K) = K$ may be interpreted as the upper bound for the number of firms, as well as the level of net worth, that the economy could ever sustain. (A2) means that the economy never runs out of the potential supply of the entrepreneurs, thus ensuring that the scarcity of the saving and of the credit, not the scarcity of the entrepreneurial talents, will drive the dynamics of business formation in this economy. (A3) may be interpreted similarly. It ensures that there are always some inactive traders in the steady state.[11] (A4) ensures that some entrepreneurs invest in equilibrium, $k_{t+1} > 0$.

[11] It turns out that dropping (A3) would not affect the results fundamentally, but drastically increase the number of the cases that need to be examined. (A2) and (A3) are introduced to remove the unwanted implication of the assumption that each agent can manage at most one project, which was made for the analytical simplicity. Both (A2) and (A3) would not be needed if the agents were allowed to invest at any scale, subject to only the minimum investment requirement. It should also be noted that these assumptions can be weakened significantly. (A2) can be replaced by $W(\min\{K, k_c\}) < \mu_1$ and (A3) by $W(k_{cc}) - k_{cc} < m\mu_2$, where k_c and k_{cc} are the values defined later.

The Investment Schedules:

Because we have set $\lambda_1 = 1$, the borrowing constraint for the entrepreneurs, (3), is never binding, whenever (1) holds, and (1) always holds because of (A4). If (1) holds with the strict inequality, all the entrepreneurs start firms. If (1) holds with the equality, they are indifferent. Therefore, the investment schedule by the entrepreneurs is given simply by the following complementarity slackness condition,

(7) $0 < k_{t+1} \leq \mu_1, \qquad \Pi(k_{t+1}) \geq r_{t+1},$

which is illustrated in Figures 1a and 1b. As shown below, (A1) and (A2) ensure that $k_{t+1} < \mu_1$ and $\Pi(k_{t+1}) = r_{t+1}$ in equilibrium. The investment demand schedule by the entrepreneurs is thus downward-sloping in the relevant range. In words, the return to business investment declines when more firms are active.

Let us now turn to the investment schedule by the traders. To this end, it is useful to define $R(W(k_t))$, the maximal rate of return that the traders could pledge to the lenders without violating the profitability and borrowing constraints, (2) and (4), which takes the following form:

$$R(W(k_t)) \equiv \begin{cases} \lambda R/[1 - W(k_t)/m] & \text{if } k_t < k_\lambda, \\ R & \text{if } k_t \geq k_\lambda, \end{cases}$$

where k_λ is uniquely given by $W(k_\lambda) \equiv (1 - \lambda)m$. That is, for $k_t < k_\lambda$, the borrowing constraint, (4), is the relevant constraint and for $k_t \geq k_\lambda$, the profitability constraint, (2), is the relevant constraint. Note that $R(W(k_t))$ is increasing in k_t for $k_t < k_\lambda$, because a higher net worth would ease their borrowing constraint, allowing them to credibly pledge a higher return to the lender. For $k_t \geq k_\lambda$, the borrowing constraint is no longer binding, hence $R(W(k_t)) = R$, independent of k_t. Thus the investment schedule by the traders may be expressed as

$$\text{(8)} \qquad mx_{t+1} \begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)), \\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)), \\ = 0 & \text{if } r_{t+1} > R(W(k_t)). \end{cases}$$

In Figures 1a and 1b, equation (8) is illustrated as a step function, which graphs $W(k_t) - mx_{t+1}$.

(A2) and (A3) are chosen simply because k_c and k_{cc} depends also on R and λ_2, so the meanings of these alternative assumptions may be less obvious to the reader.

The Credit Market Equilibrium:

The credit market equilibrium requires that r_{t+1} adjust to equate the aggregate investment and the aggregate saving, i.e., $k_{t+1} + mx_{t+1} = w_t$, or equivalently

(9) $k_{t+1} = W(k_t) - mx_{t+1}$

There are three cases to be distinguished, depending on the value of k_t.[12]

Figure 1a illustrates the case where $R(W(k_t)) < \Pi(W(k_t))$, or equivalently, $k_t < k_c$, where k_c is defined uniquely by $R(W(k_c)) \equiv \Pi(W(k_c))$. In this case, the net worth is so low that the Bad projects cannot be financed ($x_{t+1} = 0$) and all the credit go to the Good projects so that

(10a) $k_{t+1} = W(k_t) < \mu_1$ for $k_t < k_c$,

since the required rate of return is too high for the Bad projects: $r_{t+1} = \Pi(W(k_t)) > R(W(k_t))$.

Figure 1b illustrates the case where $\Pi(W(k_t)) \leq R(W(k_t)) < \Pi(W(k_t)-m\mu_2)$, or equivalently, $k_c \leq k_t < k_{cc}$, where k_{cc} is defined uniquely by $R(W(k_{cc})) \equiv \Pi(W(k_{cc})-m\mu_2)$. In this case, some but not all traders invest ($0 \leq x_{t+1} < \mu_2$). The equilibrium rate of return is equal to

(10b) $r_{t+1} = R(W(k_t)) = \Pi(k_{t+1}) = \Pi(W(k_t)-mx_{t+1})$ for $k_c \leq k_t < k_{cc}$.

An increase in k_t thus has the effect of further increasing the investment in trading. Its effect on business investment depends whether k_t is higher or less than k_λ. If $k_t > k_\lambda$, the borrowing constraint of the traders is not binding, so that the rate of return is fixed at $R(W(k_t)) = R$. Thus, the investment in the business sector remains constant at $\Pi^{-1}(R)$. On the other hand, if $k_t < k_\lambda$, the borrowing constraint for the traders is binding, so that $R(W(k_t))$ increases with k_t. A higher net worth eases the borrowing constraint of the traders, so that they can guarantee a higher rate of return to the lenders. As a result, the Good are squeezed out. In short, k_{t+1} is a decreasing function of k_t if $k_c < k_t < k_{cc}$ *and* $k_t < k_\lambda$.

Finally, there is a third case (not illustrated), where $k_t \geq k_{cc}$, or equivalently, $R(W(k_t)) \geq \Pi(W(k_t)-m\mu_2) = r_{t+1}$, hence $x_{t+1} = \mu_2$ so that

(10c) $k_{t+1} = W(k_t) - m\mu_2$ for $k_t \geq k_{cc}$.

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[12] Figures 1a-1b are drawn under the assumption, $W(k_t) < \mu_1$, which ensures $k_{t+1} < \mu_1$ in equilibrium. This assumption will be verified later. These figures are also drawn such that $W(k_t) > m\mu_2$. This need not be the case, but it does not affect the discussion in the text.

In this case, all the traders invest. Since the trading opportunities are exhausted, a further increase in the saving translates to an increase in business investment. This situation occurs as an unwanted by-product of the assumption that the traders can manage at most one trading operation, which was made to simplify the analysis of the trader's decision problem. Note, however, that we have imposed (A3) to ensure that $k_{t+1} = W(k_t) - m\mu_2 < k_t$ in this range, so that this situation would never occur in the neighborhood of the steady state.

Remark 2: A Digression on Credit Rationing: For the case shown in Figure 1b, where $r_{t+1} = \Pi(k_{t+1}) = R(W(k_t))$, only a fraction of the traders starts their operation. When $k_t \geq k_\lambda$, $r_{t+1} = R$ holds in equilibrium, and (2) is thus satisfied with equality. Some traders invest while others do not, simply because they are indifferent. When $k_t < k_\lambda$, $r_{t+1} = \lambda R/[1 - W(k_t)/m] < R$, hence (4) is binding, while (2) is satisfied with strict inequality. In other words, all the traders strictly prefer borrowing to invest over lending their net worth to others. Therefore, the equilibrium allocation necessarily involves credit rationing, whenever only a fraction of the traders starts their operation because they are denied credit. Those who denied credit cannot entice the potential lenders by promising a higher rate of return, because the lenders would know that the borrowers would not be able to keep the promise. It should be noted, however, that equilibrium credit rationing occurs in this model due to the homogeneity of the traders. Suppose instead that the traders were heterogeneous in some observable characteristics. For example, suppose each young trader receives, in addition to the labor endowment, the final goods endowment, y, which is drawn from G, a cumulative distribution function with no mass point. Then, there would be a critical level of y, $Y(w_t, r_{t+1}) \equiv m(1 - \lambda R/r_{t+1}) - w_t$, such that only the traders whose endowment income exceed $Y(w_t, r_{t+1})$ would be able to finance their investment. This makes the aggregate investment in trading, $mx_{t+1} = m[1 - G(Y(w_t, r_{t+1}))]$, smoothly decreasing in r_{t+1}, and increasing in w_t. Thus, the borrowing constraint would be enough to determine the allocation of the credit, and credit rationing would not occur.[13] What is essential for the analysis is that, when the borrowing

[13] While some authors use the term, "credit-rationing," whenever some borrowing limits exist, here it is used to describe the situation that the aggregate supply of credit falls short of the aggregate demand, so that some borrowers cannot borrow up to their borrowing limit. In other words, there is no credit rationing if every borrower can borrow up to its limit. In such a situation, their borrowing may be constrained by their net worth, which affects the borrowing limit, but not because they are credit-rationed. This is consistent with the following definition of credit rationing by Freixas and Rochet (1997, Ch.5), who attributed it to Baltensperger: "some borrower's

constraint is binding for marginal traders, an increase in the net worth of the traders increases the aggregate investment in trading, for each r_{t+1}. Therefore, it is the borrowing constraint, not the equilibrium credit rationing per se, that matters. The equilibrium credit rationing is nothing but an artifact of the homogeneity assumption, which is imposed to simplify the analysis.

The Equilibrium Trajectory:

Equations (10a)-(10c) determine k_{t+1} uniquely for each value of k_t, as follows:

$$(11) \quad k_{t+1} = \Psi(k_t) \equiv \begin{cases} W(k_t) & \text{if } k_t < k_c \\ \Pi^{-1}(R(W(k_t))) & \text{if } k_c \leq k_t < k_{cc}, \\ W(k_t) - m\mu_2 & \text{if } k_t \geq k_{cc}. \end{cases}$$

Since $k_t \leq K$ implies $k_{t+1} = \Psi(k_t) = W(k_t) - mx_{t+1} \leq W(k_t) \leq W(K) = K$, Ψ maps (0,K] into itself. Thus, for any $k_0 \in (0, K]$, this map defines a unique trajectory in (0, K]. Furthermore, $k_t \leq K$ and (A2) mean that $\mu_1 > K = W(K) \geq W(k_t)$, as has been assumed. The equilibrium trajectory of the economy can thus be solved for by applying the map (11), Ψ, iteratively, starting with the initial condition, $k_0 \in (0, K]$. This completes the description of the model.

3. The Dynamic Analysis.

We now turn to the characterization of the equilibrium dynamics. It turns out that we need to distinguish five cases, as illustrated by Figure 2a through Figure 2e.[14] What separate these cases are the relative magnitude of three critical values of k; k_c (the point at which the Bad start attracting the credit), k_λ (the point beyond which the borrowing constraint for the Bad becomes irrelevant), and W(K) = K, the maximum possible value of the net worth, as well as the stability of the steady state.

Figure 2a depicts the case where $k_c \geq K$. In this case, the Bad never attract credit and all the credit go to the Good, so that $k_{t+1} = W(k_t)$ for all $k_t \in (0, K]$. Then, from the monotonicity of

demand for credit is turned down, even if this borrower is willing to pay all the price and nonprice elements of the loan contract."

[14] Figure 2 through Figure 2e are drawn such that W(0) = 0 and W is concave. These need not be the case. (A1) assumes only that W(k) > k for all $k \in (0, K]$ and W(K) = K.

W and (A1), k_t converges monotonically to $k^* = K$ for any $k_0 \in (0,K]$. The condition, $k_c \geq K$, can be rewritten as $\Pi(K) \geq R(W(K)) = R(K)$, or equivalently

(12) $R \leq \Pi(K)\,\mathrm{Max}\{(1 - K/m)/\lambda, 1\}$.

This condition can be interpreted as follows. With a sufficiently small R, the Bad are not profitable enough to compete with the Good for the credit. When $K < m$, the condition (12) is also met with a sufficiently small λ for any R. This is because the traders must always borrow to initiate their projects with $W(k_t) \leq W(K) = K < m$. When λ is sufficiently small, they can never borrow even when $k_{t+1} = W(k_t) > \Pi^{-1}(R)$ so that the Bad are more profitable than the Good.

In the other four cases, $k_c < K$ holds, so that some traders become eventually active; $x_{t+1} > 0$ and hence $k_{t+1} < W(k_t)$ for $k_t \in (k_c, K]$. Figure 2b depicts the case, where $k_\lambda \leq k_c$ or equivalently, $W(k_c) \geq (1 - \lambda)m$, which can be rewritten as

(13) $R \leq \Pi((1 - \lambda)m)$.

Under this condition, the borrowing constraint is not binding for the traders, whenever they are active: $W(k_t) > (1 - \lambda)m$ and $R(W(k_t)) = R$ for all $k_t > k_c$. As shown in Figure 2b, the map has a flat segment, over $(k_c, \min\{k_{cc}, K\})$, but it is strictly increasing elsewhere. Furthermore, (A3) ensures $k_{cc} > W(k_{cc}) - m\mu_2$, so that the steady state is located at the flat segment.[15] The dynamics of k_t hence converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. As the business sector expands, borrower net worth improves and the profitability of business investment declines. As soon as the equilibrium rate of return drops to R, the traders start investing, because they do not face the binding borrowing constraint. Thus, the equilibrium rate of return stays constant at R, and business investment remains constant at $\Pi^{-1}(R)$.

In the three cases depicted by Figure 2c through 2e, $k_c < k_\lambda$ holds. As in Figure 2b, for $k_t > k_\lambda$, the Bad face no borrowing constraint, so that $r_t = R$ and hence $k_{t+1} = \Psi(k_t) = \Pi^{-1}(R)$. In contrast to Figure 2b, however, all these figures show the intervals below k_λ, in which $k_{t+1} = \Psi(k_t) > \Pi^{-1}(R)$ holds, suggesting an *over-investment* into the Good, $\Pi(k_{t+1}) < R$. Inside these intervals, below k_c, the saving continues to flow only into the Good; $k_{t+1} = W(k_t) > \Pi^{-1}(R)$. For $k_c < k_t <$

[15] In both Figures 3b and 3c, $k_{cc} > K$. This need not be the case, nor is it essential for the discussion in the text.

k_λ, on the other hand, the saving starts flowing into the Bad, even though they are still constrained by the low net worth, and the equilibrium rate of return remains strictly below R. Thus, we have

(14) $k_{t+1} = \Psi(k_t) = \Pi^{-1}(\lambda R/[1 - W(k_t)/m])$

for $k_c < k_t < \min\{k_\lambda, k_{cc}, K\}$. Note that (14) is decreasing in k_t. In other words, the map has a downward-sloping segment, when neither (12) nor (13) hold.

It should be clear why an increase in k_t leads to a lower k_{t+1} when the Bad are active but borrowing constrained. A higher k_t, by improving the net worth of the traders, eases their borrowing constraint, which enables them to credibly pledge a higher return to the lenders. This drives up the equilibrium rate of return. To keep the Good profitable, their investment must decline. Thus, more credit is channeled into the Bad at the expense of the Good.

Figure 2c depicts the case where the borrowing constraint for trading is not binding in the steady state. That is, the map intersects with the 45° line at a flat segment, i.e., over the interval, $(k_\lambda, \min\{k_{cc}, K\})$. The condition for this is $k_\lambda \leq k^* = \Pi^{-1}(R) < k_{cc}$. Since (A3) ensures $k^* < k_{cc}$, this occurs whenever $k_\lambda \leq \Pi^{-1}(R)$, or equivalently, $W(\Pi^{-1}(R)) \geq (1 - \lambda)m$, which can be further rewritten to

(15) $R \leq \Pi(W^{-1}((1 - \lambda)m))$.

When (15) holds but (12) and (13) are violated, the dynamics of k_t converges to $k^* = \Pi^{-1}(R) < W(k_c)$, as illustrated in Figure 2c. The dynamics is not, however, globally monotone. Starting from $k_0 < k_\lambda$, the dynamics of k_t generally overshoots k^* and approaches k^* from above.[16]

For the cases depicted by Figures 2d and 2e, (12) and (15) are both violated, which also implies the violation of (13).[17] Thus, the map intersects with the 45° line at the downward sloping part, $(k_c, \min\{k_\lambda, k_{cc}, K\})$. Therefore, the traders face the binding borrowing constraint in a neighborhood of the steady state. By setting $k_t = k_{t+1} = k^*$ in (14), the steady state is given by

(16) $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$.

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[16] The qualified "generally" is needed, because the equilibrium trajectory is monotone, if $k_0 \in \{W^{-T}(k^*) \mid T = 0, 1, 2, \ldots\}$, which is at most countable and hence of measure zero.

[17] Figures 3d and 3e are drawn such that $k_\lambda < K$. This need not be the case, nor is it essential for the discussion in the text.

Both in Figure 2d and Figure 2e, the dynamics around the steady state is oscillatory. The two figures differ in the stability of the steady state, which depends on the slope of the map at k*. Differentiating (14) and then setting $k_t = k_{t+1} = k^*$ yield,

$$\Psi'(k^*) = W'(k^*)\Pi(k^*)/\Pi'(k^*)[m - W(k^*)] = - k^*\Pi(k^*)/[m - W(k^*)],$$

where use has been made of (16) and $W'(k^*)+k^*\Pi'(k^*) = 0$. From $k^*\Pi(k^*) + W(k^*) = k^*\phi(1/k^*)$, $|\Psi'(k^*)| < 1$ if and only if

(17) $k^*\phi(1/k^*) < m$.

Note that the LHS of (17) is increasing in k*, while the LHS of (16) is decreasing in k*. Hence, (17) can be rewritten to

(18) $\lambda R > \Pi(h(m))[1 - W(h(m))/m]$,

where h(m) is defined implicitly by $h\phi(1/h) \equiv m$. This case is illustrated in Figure 2d. When (18) holds, the steady state, k*, is asymptotically stable; the convergence is locally oscillatory.

On the other hand, if

(19) $\lambda R < \Pi(h(m))[1 - W(h(m))/m]$,

then $|\Psi'(k^*)| > 1$ and hence the steady state, k*, is unstable, as illustrated in Figure 2e. For any initial condition, the equilibrium trajectory will eventually be trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$, as illustrated by the box in Figure 2e.[18] Furthermore, if $k_\lambda \geq \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions. If $k_\lambda < \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions for a generic subset of the parameter values satisfying (19) and violating (12) and (15).[19] In other words, the equilibrium dynamics exhibit permanent endogenous fluctuations almost surely.

To summarize,

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[18] In Figure 2e, $k_\lambda < W(k_c) < K < k_{cc}$. Hence, $I = [\Psi(k_\lambda), W(k_c)] = [\Pi^{-1}(R), W(k_c)]$.

[19] To see this, let $C \subset (0, K]$ be the set of initial conditions for which k_t converges. Let $k_\infty = \lim_{t\to\infty} \Psi^t(k_0)$ be the limit point for $k_0 \in C$. From the continuity of Ψ, $\Psi(k_\infty) = \lim_{t\to\infty} \Psi(k_t) = \lim_{t\to\infty} k_{t+1} = k_\infty$. Hence, $k_\infty = k^*$. Since k* is unstable, k_t cannot approach it asymptotically. It must be mapped to k* in a finite iteration. That is, there exits T such that $\Psi^T(k_0) = k^*$, or $C = \{\Psi^{-T}(k_0) | T = 0,1,2,...\}$. If $k_\lambda \geq \min\{k_{cc}, K\}$, the map has no flat segment and hence the preimage of Ψ is finite and hence C is at most countable. If $k_\lambda < \min\{k_{cc}, K\}$, the map has a flat segment, at which it is equal to $\Pi^{-1}(R)$. Thus, C is at most countable unless $\Pi^{-1}(R) \in \{\Psi^{-T}(k_0) | T = 0,1,2,...\}$, which occurs only for a nongeneric set of parameter values. (As clear from this proof, it is easy to show that, even when $k_\lambda < \min\{k_{cc}, K\}$, if $W(k_c) < \min\{k_\lambda, k_{cc}\}$, the flat segment does not belong to I. Hence, if we restrict the initial condition in I, k_t fluctuates indefinitely for almost initial conditions in I for all the parameter values satisfying (19) and violate (12) and (15).)

Proposition 1. Let $\lambda_1 = 1$ and $\lambda_2 = \lambda \in (0,1)$. Then,

A. Let $R \leq \Pi(K)\max\{(1 - K/m)/\lambda, 1\}$ or equivalently, $k_c \geq K$. Then, $x_{t+1} = 0$ and $k_{t+1} = W(k_t)$ for all $t \geq 0$.All the credit to go to the Good, with k_t converging monotonically to $k^* = K$.

B. Let $\Pi(K) < R \leq \Pi((1 - \lambda)m)$, or equivalently, $k_\lambda \leq k_c < K$. Then, k_t converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. As soon as $k_t > k_c$, some traders become active without ever being borrowing-constrained.

C. Let $\Pi((1 - \lambda)m) < R \leq \Pi(W^{-1}((1 - \lambda)m)))$ or equivalently, $k_c < k_\lambda \leq \Pi^{-1}(R)$. Then, k_t converges to the unique steady state, $k^* = \Pi^{-1}(R) < W(k_c)$. Some traders eventually become active and do not face the borrowing constraint in the neighborhood of the steady state.

D. Let $R > \Pi(W^{-1}((1 - \lambda)m)))$ and $R > \Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c , \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$. The traders face the borrowing constraint in the neighborhood of the steady state. The steady state is asymptotically stable. The convergence is locally oscillatory.

E. Let $\Pi(K)(1 - K/m)/\lambda$, $\Pi(W^{-1}((1 - \lambda)m))) < R < \Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c , \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$. The traders face the borrowing constraint in the neighborhood of the steady state. The steady state is unstable. Every equilibrium trajectory will be eventually trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$. Furthermore, the equilibrium dynamics exhibits permanent, endogenous fluctuations for almost all initial conditions.

In order to avoid a taxonomical exposition, let us focus on the case where $K < m < K\phi(1/K)$ in the following discussion.[20] Proposition 1 is illustrated by Figure 3, which divides the parameter space, (λ, R), into five regions, where Region A satisfies the conditions given in Proposition 1A, Region B satisfies those given in Proposition 1B, etc. The borders between B and C and between C and D are asymptotic to $\lambda = 1$. The borders between D and E and between A and E are hyperbolae and asymptotic to $\lambda = 0$.

If the economy is in Region A, the traders remain inactive and hence have no effect on the dynamics of business formation, and the model behaves just as the standard one-sector neoclassical growth model. There are two ways in which this could happen. First, if the trading operation is unprofitable, not surprisingly, it never competes with business investment in the credit market. More specifically, this occurs if $R \leq \Pi(K)$, i.e., when the rate of return in trading is always dominated by business investment. Second, even if $R > \Pi(K)$, so that the trading operation becomes eventually as profitable as business investment, the traders would not be able to borrow if they suffer from the severe agency problem (a small λ).

If the economy is in Region B, the trading operation eventually becomes as profitable as business investment, because $R > \Pi(K)$. Furthermore, the agency problem associated with the trading operation is so minor (λ is sufficiently high) that the traders can finance their investments as soon as the equilibrium rate of return drops to R. As a result, business investment stays constant at $\Pi^{-1}(R)$. In these cases, trading changes the dynamics of business formation, but it is simply because the credit market allocates the saving to the most profitable investments. Furthermore the dynamics always converges to the unique steady state.

The presence of the profitable trading operation has nontrivial effects on the dynamics when the economy is in Region C, D, or E, i.e., when λ is neither too high nor too low. In particular, in the cases of D and E, the traders face the binding borrowing constraint in the neighborhood of the steady state. The agency problem associated with the Bad is significant enough (i.e., λ is not too high) that the credit continues to flow into the Good, even if its rate of return is strictly less than R. Of course, the traders are eager to take advantage of the lower equilibrium rate of return, but some of them are unable to do so, because of their borrowing constraint. If λ is not too low, an improvement in net worth would ease the borrowing constraint, which drives up the equilibrium rate. This is because, with a higher net worth, they need to borrow less, and hence they would be able to guarantee the lender a higher rate of return. A rise of the equilibrium rate of return in turn causes a decline in the investment in the business sector, which reduces the net worth of the agents in the next period. When λ is relatively high (i.e., if the

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[20]Note $K < K\phi(1/K)$ for any K, because $K\phi(1/K) = K\Pi(K) + W(K) > W(K) = K$. Matsuyama (2001) offers a detailed discussion for the cases where $m < K$ and $m > K\phi(1/K)$.

economy is in Region D), this effect is not strong enough to make the steady state unstable. When λ is relatively low (i.e., if the economy is in Region E), this effect is strong enough to make the steady state unstable and generates endogenous fluctuations.[21] Thus,

<u>Corollary 1</u>

Suppose $K < m < K\phi(1/K)$. For any $R > \Pi(K)$, endogenous fluctuations occur (almost surely) for some intermediate value of λ.

This corollary is the main conclusion of the basic model. *Endogenous credit cycles occur when the Bad are sufficiently profitable (a high R) and when their agency problem is big enough that the agents cannot finance it when their net worth is low, but small enough that the agents can finance it when their net worth is high.*

Region D is also of some interest, because the local convergence toward the steady state is oscillatory, and the transitional dynamics is cyclical. If the economy is hit by recurrent shocks, the equilibrium dynamics exhibit considerable fluctuations.[22] A quick look at Proposition 1D (and Figure 3) verifies that a sufficiently high R ensures that the economy is in Region D. Thus,

<u>Corollary 2</u>.

For any $\lambda \in (0,1)$, the dynamics around the steady state is oscillatory for a sufficiently high R.

The intuition behind this result is easy to grasp. In the presence of the agency problem, the trader's borrowing constraint becomes binding, if they are sufficiently eager to invest, i.e., when the Bad are sufficiently profitable.

Note that Propositions 1D and 1E give the conditions under which the model generates locally oscillatory convergence and endogenous fluctuations for almost all initial conditions. They

[21] Technically speaking, as the economy crosses $\lambda R = \Pi(h(m))[1 - W(h(m))/m]$ from Region D to Region E, the dynamical system experiences a *flip bifurcation*.

[22] In addition, there may be endogenous fluctuations in Region D. When the parameters satisfy the conditions given in Proposition 1D, we do know that the local dynamics converges, but little can be said of the nature of global dynamics. For example, if the flip bifurcation that occurs at the boundary of D and E is of *subcritical* type,

are silent about the global dynamics. However, it is possible to show that the equilibrium trajectory may be characterized by cycles of any periods, as well as chaotic behavior, and the reader is referred to specific examples in Matsuyama (2004).

4. Reintroducing the Borrowing Constraint in the Business Sector

So far, we have analyzed the equilibrium trajectory under the assumption that $\lambda_1 = 1 > \lambda_2 = \lambda$. We are now going to show that, for any $\lambda_2 = \lambda < 1$, a small reduction in λ_1 from $\lambda_1 = 1$ would not affect the equilibrium trajectory.

Recall that the entrepreneurs start firms when both (1) and (3) are satisfied. (A4) ensures that some entrepreneurs are active, $k_{t+1} > 0$, hence both (1) and (3) hold in equilibrium. Furthermore, $k_t \leq K$ ensures that $k_{t+1} = W(k_t) - mx_{t+1} \leq W(k_t) \leq W(K) = K < \mu_1$. Therefore, at least (1) or (3) must be binding, hence

(20) $\quad \Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}$.

The credit market equilibrium is given by (8), (9) and (20). It is easy to see that, given k_t, these equations jointly determine k_{t+1} uniquely.

Let us find the condition under which the map given in eq. (11) solves the credit market equilibrium, determined by (8), (9), and (20). First, for any $k_t \geq k_c$, eq. (11) solves the credit market equilibrium if and only if the entrepreneurs do not face the binding borrowing constraint, that is, when (20) is $\Pi(k_{t+1}) = r_{t+1},$ i.e., $W(k_t) \geq 1 - \lambda_1$ for all $k_t \geq k_c$. The condition for this is $\lambda_1 \geq 1 - W(k_c)$. Then, in order for (11) to be the equilibrium, it suffices to show that $x_{t+1} = 0$ and $k_{t+1} = W(k_t)$ solve (8), (9) and (20) for $k_t < k_c$. This condition is given by

(21) $$R/\max\{[1 - W(k_t)/m]/\lambda_2, 1\} \leq \begin{cases} \lambda_1\Pi(W(k_t))/[1 - W(k_t)] & \text{if } k_t < k_{\lambda 1} \\ \Pi(W(k_t)) & \text{if } k_{\lambda 1} \leq k_t < k_c, \end{cases}$$

where $k_{\lambda 1}$ is defined implicitly by $W(k_{\lambda 1}) \equiv 1 - \lambda_1$ and satisfies $k_{\lambda 1} < k_c$. Eq. (21) is illustrated by Figure 4a (for $k_c < k_\lambda$) and Figure 4b (for $k_c > k_\lambda$). By definition of k_c, the LHS of (21) is strictly less than $\Pi(W(k_t))$ for all $k_t < k_c$. Since the RHS of (21) converges to $\Pi(W(k_t))$, as λ_1

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there are (unstable) period-2 cycles in the neighborhood of k* near the boundary on the side of Region D: see Guckenheimer and Holmes (1983, Theorem 3.5.1).

approaches one, there exists $\lambda_1' < 1$ such that eq. (21) holds for $\lambda_1 \in [\lambda_1', 1]$. Since the LHS of (21) weakly increases with λ_2, the lowest value of λ_1 for which (21) holds, λ_1', is weakly increasing in λ_2. It is also easy to see that (21) is violated for a sufficiently small λ_1, hence, $\lambda_1' > 0$. Furthermore, for any $\lambda_1 > 0$, (21) holds for a sufficiently small $\lambda_2 > 0$. Thus, λ_1' approaches zero with λ_2. One can thus conclude

<u>Proposition 2.</u>

For any $\lambda_2 = \lambda \in (0,1)$, there exists $\Lambda(\lambda_2) \in (0,1)$, such that, for $\lambda_1 \in [\Lambda(\lambda_2), 1]$, the equilibrium dynamics is independent of λ_1.[23] Λ is nondecreasing in λ_2 and satisfies $\Lambda(\lambda_2) \geq 1 - W(k_c)$, and $\lim_{\lambda 2 \to 0} \Lambda(\lambda_2) = 0$.

Proposition 2 thus means that the analysis need not be changed, as long as λ_1 is sufficiently high. In particular, Proposition 1 and their corollaries are all unaffected.

Even with a weaker condition on λ_1, the possibility of endogenous fluctuations survives. When $\lambda_1 < \Lambda(\lambda_2)$, the map depends on λ_1, but shifts continuously as λ_1 changes. Therefore, as long as the reduction is small enough, k* is unaffected and remains the only steady state of the map. Therefore, as long as $\lambda_2 = \lambda$ satisfies the condition given in Proposition 1E, the map generates endogenous fluctuations, because its unique steady state is unstable.

The above analysis thus shows that the key mechanism in generating endogenous fluctuations is that an improved economic condition eases the borrowing constraints for the Bad more than those for the Good, so that the saving is channeled into the former at the expense of the latter. The assumption made earlier that the Good faces no borrowing constraint itself is not crucial for the results obtained so far.

5. The Good, The Bad and The Ugly: Introducing Credit Multiplier

Most recent studies in macroeconomics of credit frictions, such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), have stressed a *credit multiplier* effect. An increase in net worth stimulates business investment by easing the borrowing constraint of the entrepreneurs, which further improves their net worth, leading to more business investment. This introduces *persistence* into the system. The model developed above has no such a credit multiplier effect.[24] Quite the contrary, the mechanism identified may be called a *credit reversal* effect, because an increase in net worth stimulates trading at the expense of business investment, leading to a deterioration of the net worth. This introduces *instability* into the system. Of course, these two mechanisms are not mutually exclusive. Combining the two is not only feasible but also useful because it adds some realism to the equilibrium dynamics. In an extension of the model shown below, both credit multiplier and reversal effects are present and the equilibrium dynamics exhibit persistence at a low level of economic activities and instability at a high level.[25]

The model discussed in the last section is now modified to allow the young agents to have access to a storage technology, which transforms one unit of the final good in period t into ρ units of the final good in period t+1. The storage technology is available to all the young. Furthermore, it is divisible, so that the agents can invest, regardless of their level of net worth. It is assumed that the gross rate of return on storage satisfies $\rho \in (\lambda_2 R, R)$. This restriction ensures that storage dominates trading when net worth is low, while trading dominates storage when it is high. That is, the economy now has the following three types of the investment: i) *The Good* (Business Investment), which is profitable, relatively easy to finance and generates demand for the labor endowment held by the next generation of the agents; ii) *The Bad* (Trading), which is profitable, relatively difficult to finance, and generates no demand for the labor endowment; and iii) T*he Ugly* (Storage), which is unprofitable, has no need for being financed, and generates no demand for the endowment.

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[23] The function, Λ, also depends on other parameters of the model, m, R, K, as well as the functional form of ϕ.
[24] In the model above, an increase in the net worth leads to an increase in business investment when $k_t < k_c$. This occurs because an increase in the net worth leads to an increase in the aggregate savings, all of which are used to finance the investment in the business sector. The aggregate investment in the business sector is independent of whether the entrepreneurs face the borrowing constraint. Therefore, it should not be interpreted as the credit multiplier effect.
[25] The extension presented below also serve a second purpose. In the previous models, the Bad, the only alternative to the Good, not only generates less demand spillovers but also faces tighter borrowing constraints. This might give the reader a false impression that these two features, less spillovers and tighter borrowing constraints, must go together to have instability and fluctuations. By adding a third type of projects, with less spillovers and less borrowing constraints, it shows that this is not the case. What is needed for endogenous fluctuations is that *some* profitable projects have less spillovers than others and can be financed only at a high level of economic activities.

Let s_t be the total units of the final good invested in storage at the end of period t. Then, the credit market equilibrium condition is now given by

$$(8)\qquad mx_{t+1}\begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)),\\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)),\\ = 0 & \text{if } r_{t+1} > R(W(k_t)).\end{cases}$$

$$(20)\qquad \Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}\,.$$

$$(22)\qquad s_t\ \begin{cases} = 0, & \text{if } r_{t+1} > \rho\\ \geq 0, & \text{if } r_{t+1} = \rho\\ = \infty, & \text{if } r_{t+1} < \rho\end{cases}$$

$$(23)\qquad k_{t+1} = W(k_t) - mx_{t+1} - s_t.$$

Eqs. (8) and (20) are reproduced here for easy reference. Introducing the storage technology does not make any difference in the range where $r_{t+1} > \rho$. If the storage technology is used in equilibrium, the equilibrium rate of return must be $r_{t+1} = \rho$.

Characterizing the credit market equilibrium and the equilibrium trajectory determined by (8), (20), (22) and (23) for the full set of parameter values require one to go through a large number of cases. Furthermore, in many of these cases, the presence of the storage technology does not affect the properties of the equilibrium dynamics fundamentally. In what follows, let us report one representative case, in which the introduction of the storage technology creates some important changes. More specifically, let us consider the case, where the following conditions hold. First, R and $\lambda_2 = \lambda$ satisfy the conditions given in Proposition 1E. This ensures that $k_c < k^* < k_\lambda$. Second, ρ is not too low nor too high so that $k_c < k_\rho < k^*$, where k_ρ is implicitly defined by $R(W(k_\rho)) \equiv \rho$. Third, λ_1 is large enough that $k_{\lambda 1} < k_\rho$, and small enough that the RHS of (21) is greater than ρ for $k_t < k'$ and smaller than ρ for $k_t > k'$. (It is feasible to find such λ_1 because $k_c < k_\rho$.) These conditions are illustrated in Figure 5.[26]

Then, for $k_t < k'$, the business profit is so high that all the saving goes to the investment in the business sector, and $x_{t+1} = s_t = 0$. For $k' < k_t < k_\rho$, some saving goes to the storage, $s_t > 0$, and hence $r_{t+1} = \rho > R(W(k_t))$, and the trading remains inactive, $x_{t+1} = 0$. Within this range, the borrowing constraint is binding for the entrepreneurs when $k' < k_t < k_{\lambda 1}$, and the profitability constraint is binding for the entrepreneurs when $k_{\lambda 1} < k_t < k_\rho$. For $k_\rho < k_t < \min\{k_\lambda, k_{cc}, K\}$, the storage technology is not used, $s_t = 0$. The entrepreneurs, whose borrowing constraint is not binding, compete for the credit with the traders who become active, and face the binding borrowing constraint, and the interest is given by $r_{t+1} = R(W(k_t)) > \rho$. The unstable steady state, k*, shown in Proposition 1E, is located in this range.

The equilibrium dynamics is thus governed by the following map:

$$(24)\qquad k_{t+1} = \Psi(k_t) \equiv \begin{cases} W(k_t), & \text{if } k_t \le k', \\ \Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1), & \text{if } k' < k_t \le k_{\lambda 1}, \\ \Pi^{-1}(\rho), & \text{if } k_{\lambda 1} < k_t \le k_\rho. \\ \Pi^{-1}(\lambda_2 R/[1 - W(k_t)/m]), & \text{if } k_\rho < k_t \le \min\{k_\lambda, k_{cc}\}, \\ \Pi^{-1}(R), & \text{if } k_\lambda < k_t \le k_{cc}, \\ W(k_t) - m\mu_2. & \text{if } k_t \ge k_{cc}, \end{cases}$$

where k′ is given implicitly by $\lambda_1\Pi(W(k'))/[1 - W(k')] \equiv \rho$. Eq. (24) differs from (11) for $k' < k_t < k_\rho$, where some saving go to the storage technology and the rate of return is fixed at ρ. In particular, for $k' < k_t < k_{\lambda 1}$, the investment in the business sector is determined by the borrowing constraint,

$$(25)\qquad W(k_t) = 1 - \lambda_1\Pi(k_{t+1})/\rho.$$

In this range, an increase in the net worth, $W(k_t)$, eases the borrowing constraint of the entrepreneurs, so that their investment demand goes up. Instead of pushing the equilibrium rate of return, the rise in the investment demand in the business sector is financed by redirecting the savings from storage. Intuitively enough, an increase in ρ/λ_1 shifts down the map in this range. The presence of the Ugly thus reduces the Good, which acts as a drag, slowing down the expansion processes. Unlike the Bad, however, the Ugly does not destroy the Good. And a higher business investment today leads to a higher business investment tomorrow. This mechanism is essentially identical with the one studied by Bernanke and Gertler (1989).

The crucial feature of the dynamics governed by (24) is that the credit multiplier effect is operative at a lower level of activities, while the credit reversal effect is operative at a higher level,

[26] In Figure 5, $k_{\lambda 1} < k_c$. This need not be the case, nor is it essential for the discussion in the text.

including in the neighborhood of the unstable steady state, k*. In this sense, this model is a hybrid of the model developed earlier and of a credit multiplier model à la Bernanke-Gertler.

Figure 6a illustrates the map (24) under additional restrictions, $\Psi(k_\rho) = \Pi^{-1}(\rho) \leq \min\{k_\lambda, k_{cc}\}$ and $k_{\lambda 1} > \Psi^2(k_\rho) = \Psi(\Pi^{-1}(\rho))$. The first restriction ensures that some traders remain inactive at $\Psi(k_\rho)$. This means that the trapping interval is given by $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)] = [\Psi(\Pi^{-1}(\rho)), \Pi^{-1}(\rho)]$.[27] The second restriction ensures that the trapping interval, I, overlaps with $(k', k_{\lambda 1})$, i.e., the range over which the credit multiplier effect is operative. Let us fix ρ and change λ_1. As λ_1 is reduced, $k_{\lambda 1}$ increases from $\Psi^2(k_\rho)$ to k_ρ, and at the same time, the map shifts down below $k_{\lambda 1}$. Clearly, the map has the unique steady state, k*, as long as λ_1 is not too small (or $k_{\lambda 1}$ is sufficiently close to $\Psi^2(k_\rho)$). As λ_1 is made smaller (and $k_{\lambda 1}$ approaches k_ρ), the equilibrium dynamics may have additional steady states in $(k', k_{\lambda 1})$.[28] The following proposition gives the exact condition under which that happens.

<u>Proposition 3</u>. Let k* be the (unstable) steady state in Proposition 1E.

A. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 < \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k*, two other steady states, $k_1^{**}, k_2^{**} \in (k', k_{\lambda 1})$. They satisfy $k_1^{**} < h(1) < k_2^{**}$, and k_1^{**} is stable and k_2^{**} is unstable.

B. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 = \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k*, another steady state, $k^{**} = h(1) \in (k', k_{\lambda 1})$, which is stable from below and unstable from above.

C. Otherwise, k* is the unique steady state of (24).

<u>Proof</u>. See Appendix.

If $\lambda_1 > 1 - W(h(1))$ or $\lambda_1 > \rho h(1)$, neither condition given in Proposition 3A or 3B hold, endogenous fluctuations clearly survive, because the map has a unique steady state, k*, which is unstable. Even if $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 \leq \rho h(1)$, the equilibrium dynamics may still exhibit

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[27] Note that this restriction is weaker than the restriction, $W(k_c) \leq \min\{k_\lambda, k_{cc}\}$, because $k_c < k_\rho$ implies $\Pi(W(k_c)) = R(W(k_c)) < R(W(k_\rho)) = \rho$, hence $W(k_c) > \Pi^{-1}(\rho)$.

[28] Since $k_\rho < k^* < \Pi^{-1}(\rho)$, the map does not intersect with the 45° line in $[k_{\lambda 1}, k_\rho)$.

endogenous fluctuations in $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)]$. This is because, if $h(1) < \Psi^2(k_\rho)$, $k_2^{**} < \Psi^2(k_\rho)$ as long as λ_1 is not too much lower than $\rho h(1)$, and hence the map has a unique steady state in I, k*, which is unstable, and, for any initial condition in I, the equilibrium trajectory never leaves I.

The above argument indicates that, as long as λ_1 is not too small (or ρ is not too large), the introduction of the credit multiplier effect does not affect the result that the borrowing-constrained investment in trading generates endogenous fluctuations. This does not mean, however, that the credit multiplier effect has little effects on the nature of fluctuations. The introduction of the credit multiplier effect, by shifting down the map below $k_{\lambda 1}$, can slow down an economic expansion, thereby creating asymmetry in business cycles. This is most clearly illustrated by Figure 6b, which magnifies the dynamics on the trapping interval, I, for the case where $\Psi^2(k_\rho) < h(1) < k_{\lambda 1}$. If $\lambda_1 = \rho h(1)$, as indicated in Proposition 3B, the map is tangent to the 45° line at h(1), which creates an additional steady state, k** = h(1). It is stable from below but unstable from above, and there are *homoclinic orbits*, which leave from k**, and converges to k** from below.[29] Starting from this situation, let λ_1 go up slightly. As indicated in Proposition 3C, such a change in the parameter value makes the steady state, k**, disappear, and the map is left with the unique steady state, k*, in its downward-sloping segment, which is unstable.[30] The credit multiplier effect is responsible for the segment, where the map is increasing and stays above but very close to the 45° line. Thus, the equilibrium dynamics display *intermittency*, as a tangent bifurcation eliminates the tangent point, k**, and its homoclinic orbits. The equilibrium trajectory occasionally has to travel through the narrow corridor. The trajectory stays in the neighborhood of h(1) for possibly long time, as the economy's business sector expands gradually. Then, the economy starts accelerating through the credit multiplier effect. At the peak, the traders start investing. Then, the economy plunges into a recession (possibly after going through a period of high volatility, as the trajectory oscillates around k*). Then, at the bottom, the economy begins its slow and long process of expansion. The map depicted in Figure 6b is said to display

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[29]More generally, an orbit, $\{k_t\}_{-\infty}^{+\infty}$, is *homoclinic* if there exists a periodic point, *p*, such that $\lim_{t\to\pm\infty} k_t = p$.

[30] Technically speaking, this is known as a *saddle-node* or *tangent bifurcation*.

intermittency, because its dynamic behavior is characterized by a relatively long periods of small movements punctuated by intermittent periods of seemingly random-looking movements.[31]

6. Concluding Remarks

This paper has presented dynamic general equilibrium models of imperfect credit markets, in which the economy fluctuates endogenously along its unique equilibrium path. The model is based on the heterogeneity of investment projects. In the basic model, there are two types of projects: the Good and the Bad. The Good require the inputs supplied by others. By generating demand for them, they improve net worth of other borrowers. The Bad are independently profitable, so that they generate less demand spillovers than the Good. Furthermore, the Bad are subject to the borrowing constraint so that the agents need to have a high level of net worth to be able to initiate the Bad projects. When the net worth is low, the agents cannot finance the Bad, and all the credit goes to the Good, even when the Bad are more profitable than the Good. This over-investment to the Good creates a boom, leading to an improved net worth. The agents are now able to invest into the Bad. This shift in the composition of the credit from the Good to the Bad at the peak of the boom causes a decline in net worth. The whole process repeats itself. Endogenous fluctuations occur because the Good breed the Bad and the Bad destroy the Good. An extension of the basic model introduces a third type of projects, the Ugly, which are unprofitable, contribute nothing to improve borrower net worth, but are subject to no borrowing constraint. In this extended model, when the net worth is low, the Good compete with the Ugly, which act as a drag, creating the credit multiplier effect. When the net worth is high, the Good compete with the Bad, creating the credit reversal effect. By combing the two effects, this model generates asymmetric fluctuations, along which the economy experiences a long and slow process of recovery, followed by a rapid expansion, and then, possibly after periods of high volatility, it plunges into a recession.

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[31] What is significant here is that the introduction of the credit multiplier effect can create the intermittency, regardless of the functional form of ϕ. Even without the credit multiplier effect, one can always choose a functional form of ϕ, so as to make the function $W(k) = \Psi(k)$ come close to the 45° line below k_c to generate the intermittency phenomenon. In this sense, the presence of the credit multiplier effect is not necessary for the intermittency. It simply makes it more plausible.

Several cautions should be made when interpreting the message of this paper. First, the Good (the Bad) are defined as the profitable investment projects that contribute more (less) to improve the net worth of the next generation of the agents. These effects operate solely through changes in the competitive prices. They are based entirely on pecuniary externalities, not on technological externalities. Therefore, one should not interpret a shift of the credit from the Good to the Bad as a sign of inefficiency. Of course, more credit to the Bad means bad news for the next generation of the agents, but it is also a consequence of good news for the current generation of the agents, i.e., their net worth is high.

Second, one should not hold the Bad solely responsible for credit cycles. True, the presence of the Bad is essential for credit cycles. If the Bad were removed from the models (or if they were made irrelevant by reducing R or λ so as to move the economy from Region E to Region A of Figure 3), the dynamics monotonically converges, as in the standard neoclassical growth model. Furthermore, the credit reversal takes place when the saving begins to flow into the Bad. However, it is misleading to say that the credit extended to the Bad is the cause of credit cycles. This is because credit cycles can be eliminated also if more credit were extended to the Bad. Recall that, if the agency cost associated with the Bad is made sufficiently small (a large λ), the economy moves from Region E to Region B in Figure 3. One reason why endogenous fluctuations occur in Region E is that the agency cost associated with the Bad is large enough that the saving continues to flow into the Good, even after the profitability of the Good becomes lower than that of the Bad. Without this over-investment into the Good, there would not be a boom. And without the boom that precedes it, the credit reversal could not happen. Viewed this way, one might be equally tempted to argue that the credit extended for the Good is the cause of credit cycles. It is more appropriate to interpret that the heterogeneity of the investment projects and the changing composition of the credit are the causes of credit cycles, and it should not be attributed solely to the credit extended for the Good nor to the credit extended for the Bad.

Third, even though the credit market frictions play a critical role in generating credit cycles, our analysis does not suggest that economies with less developed financial markets are more vulnerable to instability. As shown in Figure 3, endogenous cycles occur for an intermediate range of the credit market imperfections. Thus, an improvement in the credit market could introduce instability into the system. Nor should one conclude that a significant improvement in the credit market could eliminate endogenous cycles. In the formal analysis, we have assumed that there is one type of the Bad projects, but this was only for the convenience. In reality, there might be arbitrarily many types of the Bad projects, and each type could generate instability for a different range of the credit market imperfection. Then, any further improvement in the credit market may simply replace some types of the Bad projects by other types of the Bad projects, in which case instability would never be eliminated.

Fourth, by demonstrating recurrent fluctuations through the iterations of the time-invariant deterministic nonlinear maps, this paper is not trying to argue that exogenous shocks are unimportant to understanding economic fluctuations. What it suggests is that exogenous shocks do not need to be large,--indeed, they can be arbitrarily small--, to generate large fluctuations. It would be interesting to extend the model to add some exogenous shocks and investigate the interplay between the shocks and internal destabilizing mechanism of the nonlinear system. For example, consider adding some exogenous recurrent technology shocks to the final goods production, which affects the profitability of the Good projects. Imagine, in particular, such an extension in the hybrid model of Section 5. That would shake the nonlinear map of eq. (24) up and down. Suppose that, for most of the times, the shocks are so small that the map satisfies the condition given in Proposition 3A, so that the equilibrium dynamics oscillate around the unique stable steady state, k_1^{**}, and hence can be described by the credit multiplier model a la Bernanke-Gertler. Every once in a while, the shocks are just large enough to push up the map so that it briefly satisfies the condition given in Proposition 3C. Then, after such shocks, the economy experiences a rapid expansion, and possibly after a period of high volatility, plunges into a recession, from which the economy recovers slowly to the old steady state. Such an extension may be useful for understanding why credit market frictions, while introducing persistence into the investment dynamics most of the times, also make the economy subject to intermittent episodes of “mania, panics, and crashes,” as described in Kindleberger, without relying on any irrationality.

Appendix: Proof of Proposition 3.

Because the introduction of the storage technology changes the map only for (k', k_ρ), and since $k_\rho < k^* < \Pi^{-1}(\rho)$ implies $\Psi(k_t) > k_t$ in $[k_{\lambda 1}, k_\rho)$, the dynamical system, (24), could have additional steady states only in $(k', k_{\lambda 1})$, where it is given by

(*) $\quad k_{t+1} = \Psi(k_t) = \Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1)$.

By differentiating (*) and then setting $k_t = k_{t+1} = k^{**}$, the slope of the map at a steady state in this range is equal to $\Psi'(k^{**}) = -\rho W'(k^{**})/\Pi'(k^{**})\lambda_1 = \rho k^{**}/\lambda_1$, which is increasing in k^{**}. Since Ψ is continuous, and $\Psi(k') > k'$ and $\Psi(k_{\lambda 1}) > k_{\lambda 1}$ hold, this means that either

i) the map intersects with the 45° line twice at k_1^{**} and $k_2^{**} > k_1^{**}$;

ii) it is tangent to the 45° line at a single point, $k^{**} \in (k', k_{\lambda 1})$ and $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})/\{k^{**}\}$; or

iii) $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})$.

Consider the case of ii). Then, $\rho k^{**}/\lambda_1 = 1$ and $k^{**} = \Pi^{-1}(\rho[1 - W(k^{**})]/\lambda_1)$, which imply that $\Pi(k^{**})k^{**} + W(k^{**}) = \phi(1/k^{**})k^{**} = 1$, or $k^{**} = h(1) = \lambda_1/\rho$. Thus, $\lambda_1 = \rho h(1)$ implies that (*) is tangent to the 45° line at $k^{**} = h(1)$. Furthermore, $h(1) = \Psi(h(1)) < W(h(1))$ implies that $\lambda_1\Pi(W(h(1)))/[1 - W(h(1))] < \lambda_1\Pi(h(1))/[1 - W(h(1))] = \lambda_1/h(1) = \rho = \lambda_1\Pi(W(k'))/[1 - W(k')]$, or equivalently, $k^{**} = h(1) > k'$, and $\lambda_1 < 1 - W(h(1))$ implies that $k^{**} = h(1) < k_{\lambda 1}$. This proves Proposition 3B. The case of i) can always be obtained by increasing ρ from the case of i), which shifts down the map to create a stable steady state at $k_1^{**} < h(1)$ and an unstable steady state at $k_2^{**} > h(1)$. This proves Proposition 3A. Otherwise, iii) must hold, i.e., the map must lie above 45° line over the entire range, in $(k', k_{\lambda 1})$, which completes the proof of Proposition 3.

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